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PRODUCTS OF DECOMPOSITIONS OF E^n

BY

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ABSTRACT. In this paper we give a sufficient condition for the existence of a homeomorphism $h: E^m/G \times E^n/H \to E^{m+n}$, where G and H are u.s.c. decompositions of Euclidean space. This condition is then shown to hold for a wide class of examples in which the decomposition spaces E^m/G and E^n/H may fail to be Euclidean.

It is well known that manifolds can be written as the product of topological spaces which may themselves fail to be manifolds. In [4], Bing gives a factorization of Euclidean 4-space, $E^4 \cong E^3/G \times E^1$, in which the factor E^3/G is not Euclidean. This factor is the so-called dogbone decomposition of E^3. In [2], Andrews and Curtis give a simpler example in which the collection G consists of a single arc. This example was generalized in [8] to give the following example: for arcs $\alpha \subset E^m$ and $\beta \subset E^n$ it is true that $E^m/\alpha \times E^n/\beta \cong E^{m+n}$. Hence for badly embedded arcs α and β, there exist factorizations of E^{m+n}, neither factor being Euclidean. It is the purpose of this paper to show that this phenomenon occurs for fairly general decompositions of E^m and E^n. In particular if G is a decomposition of E^m and if H is a decomposition of E^n, then relatively mild conditions on G and H imply that

$$E^m/G \times E^n/H \cong E^{m+n}.$$

By relatively mild it is meant only that G and H satisfy certain conditions which many examples in the literature are known to possess.

The term decomposition will always mean a monotone, upper-semicontinuous decomposition. If G is a decomposition of E^m, then H_G denotes the union of the nondegenerate elements of G and E^m/G denotes the decomposition space of G. Notice that the nondegenerate elements of a decomposition form an upper-semicontinuous collection of sets in E^n. For the moment it is such u.s.c. collections which we investigate.

Let $A = \{\alpha\}$ be a collection of continua in E^m. For convenience we will let $A^* = \bigcup \{\alpha \mid \alpha \in A\}$. If $A = \{\alpha\}$ and $B = \{\beta\}$ are collections of continua with

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$A^* \subset E^m$ and $B^* \subset E^n$, then let $A \times B = \{\alpha \times \beta | \alpha \in A \text{ and } \beta \in B\}$, which is a collection in $E^m \times E^n$. For example, if $A^* \subset E^m$ and if we think of E^n as a collection of points, then

$$A \times E^n = \{\alpha \times w | \alpha \in A \text{ and } w \in E^n\}.$$

Let $G = \{\gamma\}$ be a collection of continua, $G^* \subset E^n$. Then the collection G is said to be *shrinkable* if and only if for each $\epsilon > 0$ there exists a homeomorphism $h_\epsilon: E^n \to E^n$ such that

(1) $h = 1$ outside $N_\epsilon(G^*)$, the ϵ-neighborhood of G^*,

(2) for each $\gamma \in G$, diameter $h(\gamma) < \epsilon$,

(3) for each $\gamma \in G$, there exists $\gamma' \in G$ such that $\gamma \cup h(\gamma) \subset N_\epsilon(\gamma')$, and

(4) for each point $p \in E^n$, either $h(p) = p$, or $p \cup h(p) \subset N_\epsilon(\gamma)$ for some $\gamma \in G$.

In the sequel we shall always use the notation h_ϵ, f_δ, etc., to denote a homeomorphism as above, satisfying conditions (1)–(4) with respect to the number ϵ, δ, etc. These maps will be referred to as *shrinking homeomorphisms.*

Generally we use the shrinkability of a collection G in the following manner. Inductively we define a sequence of shrinking homeomorphisms, take their composition and passing to the limit we get a map $h: E^n \to E^n$. If the point-inverses of h coincide with the point-inverses of the quotient map $\pi: E^n \to E^n/G$, then the composition $h \cdot \pi^{-1}$ is continuous. A necessary and sufficient condition that $h \cdot \pi^{-1}$ be a homeomorphism is that h be a *proper* map, i.e., $h^{-1}(K)$ is compact whenever K is compact.

In order to prove that h is well-behaved, it is frequently the case that the shrinking homeomorphisms are uniformly continuous ([4], [6]). McAuley defines shrinkability in [12], and includes a uniform property which helps assure convergence of sequences of shrinking homeomorphisms. However, there are examples due to Andrews and Rubin [14] in which such nice shrinking maps could not be produced. In their examples (and in Proposition 1 below), it is the nature of the collection which allows one to prove convergence.

Proposition 1. *Let A and B be u.s.c. collections of compact continua, A^* and B^* compact, contained in E^m and E^n respectively. If the collection*

$$D = \{A \times B\} \cup \{A \times (E^n - B^*)\} \cup \{(E^m - A^*) \times B\}$$

is shrinkable, then $E^m/A \times E^n/B \cong E^{m+n}$.

Proof. For elements of the form $(\alpha \times x) \in A \times (E^n - B^*)$, and any shrinking homeomorphism f_ϵ, if $(\alpha \times x)$ lies outside $N_\epsilon(A \times B)$ then

$$(\alpha \times x) \cup f_\epsilon(\alpha \times x) \subset N_\epsilon(\alpha' \times x') \subset N_{2\epsilon}(\alpha' \times x)$$

for some $\alpha' \in A$ and $x' \in (E^n - B^*)$. So we adjust our notation so that given any shrinking homeomorphism f_ϵ and $(\alpha \times x)$ outside of $N_\epsilon(A \times B)$, $(\alpha \times x) \cup f_\epsilon(\alpha \times x) \subset N_\epsilon(\alpha' \times x)$ for some $\alpha' \in A$. Similarly, $(y \times \beta) \cup f_\epsilon(y \times \beta) \subset N_\epsilon(y \times \beta')$ for some $\beta' \in B$. For points $(x \times y)$ not in $N_\epsilon(A \times B)$ we require that either

$f_\epsilon|(x \times y) = 1$, or

$(x \times y) \cup f_\epsilon(x \times y) \subset N_\epsilon(\alpha \times y)$ for some $\alpha \in A$, or

$(x \times y) \cup f_\epsilon(x \times y) \subset N_\epsilon(x \times \beta)$ for some $\beta \in B$.

Now let $\sum_{i=1}^{\infty} \epsilon_i < \frac{1}{2}$. Let $0 < \delta_1 < \epsilon_1$ and $h_1 = f_{\delta_1}$, where f_{δ_1} shrinks D according to the conventions above. Let $K_1 = [-1, 1]^{m+n}$ and suppose (without loss of generality) that $N_{1/2}(A \times B) \subset K_1$. Let $K_2 = [-2, 2]^{m+n}$ and note that there exists $\delta_2 < \min\{\epsilon_2, \delta_1\}$ such that if $X \subset K_2$ and diameter $X < \delta_2$, then diameter $h_1(X) < \delta_1$. Let $h_2 = h_1 f_{\delta_2}$.

Inductively we suppose that δ_i and h_i have been defined. Let $K_{i+1} = [-i-1, i+1]^{m+n}$ and choose $\delta_{i+1} < \min\{\epsilon_{i+1}, \delta_i\}$ such that if $X \subset K_{i+1}$ and diameter $X < \delta_{i+1}$, then diameter $h_i(X) < \delta_i$. Let $h_{i+1} = h_i f_{\delta_{i+1}}$.

Let $h = \lim_{j\to\infty} h_j$. To see that h is well defined and continuous, let $y \in D$; say $y \subset K_N$. Since each f_{δ_j} cannot move points out toward infinity more than δ_j, it follows that $h_j(y) \subset N_{1/2}(K_N)$ for all j. For $j > N$, diameter $h_{j+1}(y) < \delta_j$. If $p \in y$ is a representative point of $D^* \subset E^{m+n}$, we have

$$h_{j-1}(y) \cup h_j(y) \subset h_{j-1}(N_{\delta_j}(y_j)) \subset N_{\delta_{j-1}}(h_{j-1}(y_j)),$$

where $y_j \in D$ and j is large. Therefore,

$$\text{distance}(h_{j-1}(p), h_j(p)) < \delta_{j-2} + 2\delta_{j-1}.$$

For points $p \notin D^*$, the sequence $\{h_j(p)\}$ is eventually constant. In any case, for sufficiently large integers r and s,

$$\text{distance}(h_r(p), h_s(p)) < 3 \sum_{i=r-2}^{s-1} \delta_i.$$

Therefore h is well defined and continuous.

We have seen that, for points $p \in K_N$, $f_{\delta_j}^{-1}(p) \in N_{\delta_j}(K_N)$, and so $h_j^{-1}(p) \in N_{1/2}(K_N)$. Let $p_j = h_j^{-1}(p)$; since $\{p_j\} \subset K_{N+1}$, there is a limit point p' and $h(p') = p$. This shows that h is an epimorphism.

In a similar fashion, $h^{-1}(C)$ is bounded whenever $C \subset E^{m+n}$ is compact. Thus h is a proper map. The elements of D are shrunk to points, with different elements of D going to different points. Therefore $E^m/A \times E^n/B \cong E^{m+n}$.

Recall that a continuum $X \subset E^n$ is said to have *property* UV^∞ if for every open set U containing X, there exists an open set V, $X \subset V \subset U$, such that the inclusion $i: V \to U$ is null homotopic. This is really a property of the embedding of X, but is a topological property of X when we restrict our attention to embeddings in ANR's (see [9]).

Let X be a compactum in the interior of a topological n-manifold M. We say that X is *definable by cells* in M if there is a sequence $\{B_i\}_1^\infty$ where each B_i consists of a finite number of disjoint n-cells in M, with $B_{i+1} \subset \operatorname{Int} B_i$ for each i and $X = \bigcap_{i=1}^\infty B_i$. The set X is said to be *cellular* if it is connected and definable by cells.

Theorem 1. *Suppose* $\alpha \subset E^m$ *and* $\beta \subset E^n$ *are compact,* UV^∞ *continua such that* $\alpha \times E^n$ *and* $\beta \times E^m$ *are shrinkable. Then*

$$E^m/\alpha \times E^n/\beta \cong E^{m+n}.$$

This theorem follows directly from Theorem 2, but an independent proof is simpler and gives some insight into the proof of Theorem 2. The interested reader can supply the appropriate ϵ's and δ's in the following outline.

Outline of Theorem 1. From Theorem 8 in [13] and observations on these proofs made in [11], it is easy to see that $\alpha \times \beta$ is cellular in $E^m \times E^n$. Using this fact, one can construct a uniformly continuous map which shrinks $\alpha \times \beta$ to a point, is a homeomorphism off of $\alpha \times \beta$, and is the identity outside an arbitrary preassigned neighborhood of $\alpha \times \beta$. If we call this map f, then it is well known that the image of f is homeomorphic to $E^m \times E^n$.

Now the hypothesis gives shrinking homeomorphisms f' of $\alpha \times E^n$ and f'' of $E^m \times \beta$. Using Theorem 7.1 of [5], these are replaced by homeomorphisms f_1 and f_2 which are the identity near $\alpha \times \beta$, but agree with f' and f'' respectively outside a small neighborhood of $\alpha \times \beta$.

Consider the composition ff_1f_2, which shrinks $\alpha \times \beta$ to a point and shrinks $\{\alpha \times w \mid w \in E^n - \beta\}$ and $\{z \times \beta \mid z \in E^m - \alpha\}$ to small sets. By passing to the limit of a sequence of such maps that shrink things smaller and smaller, we verify the conclusion of Theorem 1.

Theorem 2. *Let* $A = \{\alpha\}$ *and* $B = \{\beta\}$ *be upper-semicontinuous collections of compact continua such that* $A^* \subset E^m$, $B^* \subset E^n$, A^* *and* B^* *compact,* $A \times E^n$ *is shrinkable,* $B \times E^m$ *is shrinkable, and* $A \times B$ *is shrinkable. Then the collection*

$$(A \times B) \cup (A \times (E^n - B^*)) \cup ((E^m - A^*) \times B)$$

is shrinkable.

If we denote by G_A and G_B the decompositions of E^m and E^n whose nondegenerate elements consist of the elements of A and B respectively, then we get the following

Corollary. *Under the hypothesis of Theorem* 2,

$$E^m/G_A \times E^n/G_B \cong E^{m+n}.$$

In essence, Theorem 2 follows from the u.s.c. conditions on A and B. The proof involves several lemmas, all of which assume the hypotheses of Theorem 2. The lemmas simply state things about u.s.c. collections; the proofs are similar, so most are omitted.

Lemma 1. *Given* $\epsilon > 0$, *there exists* δ', $0 < \delta' < \epsilon$, *such that if* $0 < \delta \leq \delta'$, *then any* f_δ *shrinking* $A \times B$ *satisfies the following: for all* $\alpha \in A$, $x \in E^n$, *either*

(1) $f_\delta | (\alpha \times x) = 1$, *or*

(2) $(\alpha \times x) \cup f_\delta(\alpha \times x) \subset N_\epsilon(\alpha' \times \beta')$ *for some* $\alpha' \in A$, $\beta' \in B$.

Lemma 2. *Given* $\epsilon > 0$, *there exists a* $\delta' > 0$ *such that if* $0 < \delta \leq \delta'$ *then there is a* $\gamma > 0$ *such that for all* $\alpha \in A$, $\beta \in B$ *and* f_δ *shrinking* $A \times B$, *there exist* $\alpha' \in A$ *and* $\beta' \in B$ *with*

$$N_\gamma(\alpha \times \beta) \cup f_\delta(N_\gamma(\alpha \times \beta)) \subset N_\epsilon(\alpha' \times \beta').$$

Lemma 3. *Given* $\epsilon > 0$ *and* $\beta \in B$, *there exists a* $\delta'_\beta > 0$ *such that* $0 < \delta \leq \delta'_\beta$ *implies that any* f_δ *shrinking* $A \times E^n$ *satisfies the following: for all* $\alpha \in A$, *there exists* $\alpha' \in A$ *such that*

$$(\alpha \times \beta) \cup f_\delta(\alpha \times \beta) \subset N_\epsilon(\alpha' \times \beta).$$

Lemma 4. *Given* $\epsilon > 0$, *there exists a* $\delta' > 0$ *such that* $0 < \delta \leq \delta'$ *implies that any* f_δ *shrinking* $A \times E^n$ *satisfies the following: for all* $\alpha \in A$, $\beta \in B$, *there exist* $\alpha' \in A$ *and* $\beta' \in B$ *such that*

$$(\alpha \times \beta) \cup f_\delta(\alpha \times \beta) \subset N_\epsilon(\alpha' \times \beta').$$

Lemma 5. *Given* $\epsilon > 0$, *there exists a* $\delta' > 0$ *such that* $0 < \delta \leq \delta'$ *implies that any* f_δ *shrinking* $A \times E^n$ *satisfies the following: for all* $\beta \in B$ *and* $x \in E^m$, *either*

(1) $f_\delta | (x \times \beta) = 1$, *or*

(2) $(x \times \beta) \cup f_\delta(x \times \beta) \subset N_\epsilon(\alpha' \times \beta')$ *for some* α', β'.

Lemma 6. *Given* $\epsilon > 0$ *there exists a* $\delta' > 0$ *such that* $0 < \delta \leq \delta'$ *implies the existence of a* $\gamma > 0$, γ *depends on* δ, *such that for any* f_δ *shrinking* $A \times B$, *either*

$f_\delta | N_\gamma(\alpha \times x) = 1$, *or*

$N_\gamma(\alpha \times x) \cup f_\delta(N_\gamma(\alpha \times x)) \subset N_\epsilon(\alpha' \times \beta')$.

Lemma 7. *Given* $\epsilon > 0$ *there exists a* $\delta' > 0$ *such that* $0 < \delta \leq \delta'$ *implies the existence of a* $\gamma > 0$, $\gamma = \gamma(\delta)$, *such that for any* f_δ *shrinking* $E^m \times B$,

(1) $N_\gamma(y \times \beta) \cup f_\delta(N_\gamma(y \times \beta)) \subset N_\epsilon(y \times \beta')$,
(2) $N_\gamma(\alpha \times \beta) \cup f_\delta(N_\gamma(\alpha \times \beta)) \subset N_\epsilon(\alpha'' \times \beta'')$,
(3) $f_\delta | N_\gamma(\alpha \times x) = 1$, *or* $N_\gamma(\alpha \times x) \cup f_\delta(N_\gamma(\alpha \times x)) \subset N_\epsilon(\alpha' \times \beta')$.

Proof of Lemma 1. Suppose not; then for each $\delta' = 1/n$ $(1/n < \epsilon)$, there exist a $\delta_n \leq 1/n$ and $\alpha_n \in A$, $x_n \in E^n$ such that

(1) $f_{\delta_n} | (\alpha_n \times x_n) \neq 1$, and
(2) $(\alpha_n \times x_n) \cup f_{\delta_n}(\alpha_n \times x_n) \not\subseteq N_\epsilon(\alpha \times \beta)$ for any $\alpha \in A$, $\beta \in B$.

Condition (1) assures us that all the $(\alpha_n \times x_n)$ are in some compact neighborhood of $A \times B$. Since A^* is compact, let $a_n \in \alpha_n$, for each n, and it follows that $\{a_n\}$ has a limit point, say a. By passing to a subsequence we may suppose that $a \in \underline{\lim}\, \alpha_n$. Since A^* is compact, $a \in \alpha$ for some $\alpha \in A$. Since A is u.s.c., it follows that $\overline{\lim}\, \alpha_n \subset \alpha$.

Passing to subsequences when necessary, we may suppose, w.l.o.g., that $\{x_n\}$ converges to x. Condition (1) provides that $x \in \beta$ for some $\beta \in B$. For otherwise, there exists $N > 0$ such that $N_{\delta_N}(\alpha \times x) \cap N_{\delta_N}(A \times B) = \emptyset$. Thus for sufficiently large n, $f_{\delta_n} | (\alpha_n \times x_n) = 1$. So let $x \in \beta \in B$.

Consider $\alpha \times \beta$. For each n, there exists $\alpha_n' \in A$ and $\beta_n' \in B$ such that $(\alpha \times \beta) \cup f_{\delta_n}(\alpha \times \beta) \subset N_{1/n}(\alpha_n' \times \beta_n')$. Evidently $\underline{\lim}\, \alpha_n' \neq \emptyset$ and $\overline{\lim}\, \alpha_n' \subset \alpha$. Similarly $\overline{\lim}\, \beta_n' \subset \beta$.

Hence there exists N such that $n > N$ implies that

(1) $\alpha_n \times x_n \subset N_\epsilon(\alpha \times \beta)$, and
(2) $N_{1/n}(\alpha_n' \times \beta_n') \subset N_\epsilon(\alpha \times \beta)$.

Hence $(\alpha_n \times x_n) \cup f_{\delta_n}(\alpha_n \times x_n) \subset N_\epsilon(\alpha \times \beta)$. From this contradiction, the lemma follows.

Proof of Theorem 2. Let $\epsilon > 0$ be given. A homeomorphism $h\colon E^m \times E^n \to E^m \times E^n$ which shrinks the collection

$$\{A \times B\} \cup \{A \times (E^n - B^*)\} \cup \{(E^m - A^*) \times B\}$$

is constructed as follows: First shrink $A \times B$. Select $\delta' < \epsilon$ satisfying Lemmas 1, 2, and 6. Let $\delta_1 \leq \delta'$ be positive. Using Lemmas 2 and 6 we can choose $\delta_2 > 0$ such that

$$N_{\delta_2}(\alpha \times \beta) \cup f_{\delta_1}(N_{\delta_2}(\alpha \times \beta)) \subset N_\epsilon(\alpha' \times \beta'),$$

either $f_{\delta_1} | N_{\delta_2}(\alpha \times x) = 1$, or

$$N_{\delta_2}(\alpha \times x) \cup f_{\delta_1}(N_{\delta_2}(\alpha \times x)) \subset N_\epsilon(\alpha'' \times \beta'')$$

and either $f_{\delta_1} | N_{\delta_2}(y \times \beta) = 1$, or

$$N_{\delta_2}(y \times \beta) \cup f_{\delta_1}(N_{\delta_2}(y \times \beta)) \subset N_\epsilon(\alpha''' \times \beta''').$$

Since f_{δ_1} is the identity off of a compact set, it is uniformly continuous, so we may impose additional requirements on δ_2. We require that diameter $X < \delta_2$ imply that diameter $f_{\delta_1}(X) < \delta_1$, and diameter $f_{\delta_1}(N_{\delta_2}(\alpha \times \beta)) < \delta_1$ for all $\alpha \times \beta$ in $A \times B$.

Now shrink $E^m \times B$. In Lemmas 3–5 and 7, let δ_2 be used as the $\epsilon > 0$, and select δ' which works for all these lemmas. Let $\delta_3 < \delta'$ be positive, and select $\delta_4 > 0$ (by Lemma 7) such that

$N_{\delta_4}(y \times \beta) \cup f_{\delta_3}(N_{\delta_4}(y \times \beta)) \subset N_{\delta_2}(y \times \beta')$,

$f_{\delta_3} | N_{\delta_4}(\alpha \times x) = 1$, or

$N_{\delta_4}(\alpha \times x) \cup f_{\delta_3}(N_{\delta_4}(\alpha \times x)) \subset N_{\delta_2}(\alpha'' \times \beta'')$, and

$N_{\delta_4}(\alpha \times \beta) \cup f_{\delta_3}(N_{\delta_4}(\alpha \times \beta)) \subset N_{\delta_2}(\alpha''' \times \beta''')$.

Let K be a compact neighborhood containing $N_{2\epsilon}(A \times B)$ and such that $f_{\delta_1} | (E^m \times E^n) - K = 1$. We impose an additional requirement on δ_4. If $X \subset K$ and diameter $X < \delta_4$, then diameter $f_{\delta_3}(X) < \delta_3$.

Now shrink $A \times E^n$ by a shrinking homeomorphism f_{δ_4}. We see that

$(\alpha \times x) \cup f_{\delta_4}(\alpha \times x) \subset N_{\delta_4}(\alpha' \times x)$,

$(\alpha \times \beta) \cup f_{\delta_4}(\alpha \times \beta) \subset N_{\delta_4}(\alpha'' \times \beta'')$,

$f_{\delta_4} | y \times \beta = 1$, or

$(y \times \beta) \cup f_{\delta_4}(y \times \beta) \subset N_{\delta_4}(\alpha''' \times \beta''')$.

Set $h = f_{\delta_1} f_{\delta_3} f_{\delta_4}$; we must verify that h satisfies conditions 1–4 of shrinkability. Condition 1 is easy to verify. To check conditions 2 and 3, we use the following diagram to help enumerate the various possibilities.

$$f_{\delta_4} \xrightarrow[\text{II miss}]{\text{I hit}} f_{\delta_3} \xrightarrow[\text{IV miss}]{\text{III hit}} f_{\delta_2} \xrightarrow[\text{VI miss}]{\text{V hit}} .$$

We will use case numbers like Case I-III-V so show how the elements of the collection $\{A \times B\} \cup \{A \times (E^n - B^*)\} \cup \{(E^m - A^*) \times B\}$ are affected by h.

We consider cases for elements of the form $(\alpha \times x)$.

Case I-III-V.

$(\alpha \times x) \cup f_{\delta_4}(\alpha \times x) \subset N_{\delta_4}(\alpha' \times x)$ and diameter $f_{\delta_4}(\alpha \times x) < \delta_4$.

$N_{\delta_4}(\alpha' \times x) \cup f_{\delta_3}(N_{\delta_4}(\alpha' \times x)) \subset N_{\delta_2}(\alpha'' \times \beta'')$ and diameter $f_{\delta_3} f_{\delta_4}(\alpha \times x) < \delta_3$,

$N_{\delta_2}(\alpha'' \times \beta'') \cup f_{\delta_1}(N_{\delta_2}(\alpha'' \times \beta'')) \subset N_\epsilon(\alpha''' \times \beta''')$.

Therefore $(\alpha \times x) \cup h(\alpha \times x) \subset N_\epsilon(\alpha''' \times \beta''')$ and diameter $h(\alpha \times x) < \delta_1$.

Case I-IV-V.

$(\alpha\times x)\cup f_{\delta_4}(\alpha\times x)\subset N_{\delta_4}(\alpha'\times x)$ and diameter $f_{\delta_4}(\alpha\times x)<\delta_4$, $f_{\delta_3}|N_{\delta_4}(\alpha'\times x)=1$ and hence diameter $f_{\delta_3}f_{\delta_4}(\alpha\times x)<\delta_2$. Therefore $(\alpha\times x)\cup h(\alpha\times x)\subset f_{\delta_1}(N_{\delta_4}(\alpha'\times x))\subset N_\epsilon(\alpha''\times\beta'')$. Also diameter $h(\alpha\times x)=$ diameter $f_{\delta_1}f_{\delta_3}f_{\delta_4}(\alpha\times x)<\delta_1$.

Case I-IV-VI.

$(\alpha\times x)\cup h(\alpha\times x)\subset N_{\delta_4}(\alpha'\times x)$ and diameter $h(\alpha\times x)=$ diameter $f_{\delta_4}(\alpha\times x)<\delta_4$.

Case I-III-VI.

Is impossible, as are all the cases which begin with II.

Consider cases for elements of the form $y\times\beta$. These are similar to the cases above. For example:

Case II-III-V.

$f_{\delta_4}|y\times\beta=1$,

$(y\times\beta)\cup f_{\delta_3}f_{\delta_4}(y\times\beta)\subset N_{\delta_2}(y\times\beta')$ and diameter $f_{\delta_3}(y\times\beta)<\delta_3$,

$(y\times\beta)\cup f_{\delta_1}f_{\delta_3}f_{\delta_4}(y\times\beta)=(y\times\beta)\cup h(y\times\beta)\subset f_{\delta_1}(N_{\delta_2}(y\times\beta'))\subset N_\epsilon(\alpha''\times\beta'')$ and diameter $h(y\times\beta)<\delta_1$.

For elements of the form $\alpha\times\beta$, there is only one possibility:

Case I-III-V.

$(\alpha\times\beta)\cup f_{\delta_4}(\alpha\times\beta)\subset N_{\delta_4}(\alpha'\times\beta')$,

$N_{\delta_4}(\alpha'\times\beta')\cup f_{\delta_3}(N_{\delta_4}(\alpha'\times\beta'))\subset N_{\delta_2}(\alpha''\times\beta'')$, and $N_{\delta_2}(\alpha''\times\beta'')\cup f_{\delta_1}(N_{\delta_2}(\alpha''\times\beta''))\subset N_\epsilon(\alpha'''\times\beta''')$.

Therefore $(\alpha\times\beta)\cup h(\alpha\times\beta)\subset N_\epsilon(\alpha'''\times\beta''')$ and since $h(\alpha\times\beta)\subset f_{\delta_1}(N_{\delta_2}(\alpha''\times\beta''))$, we see that diameter $h(\alpha\times\beta)<\delta_1$.

Thus h satisfies conditions 1, 2, and 3 of shrinkability, and condition 4 follows in a fashion similar to condition 3.

We turn now to applications of Theorem 2. The literature contains many examples of collections A which satisfy the condition that $A\times E^1$ is shrinkable. This is the content of [2] and [6] for examples in which A consists of a single arc or a k-cell. Bing established this shrinkability criterion for the nondegenerate elements of the dogbone decomposition of E^3 in [4]. So, much of the hypothesis of Theorem 2 is readily satisfied. The difficulty lies in establishing the shrinkability of $A\times B$, but there are sufficient conditions for this shrinkability. The most obvious are if $A\times B$ is cellular, or if $A\times B$ is simply definable by cells.

Recall that a decomposition means a monotone, u.s.c. decomposition, and if

G is a decomposition of E^n, then H_G denotes the union of the nondegenerate elements of G. The decomposition G is said to be *compact* if $\mathrm{Cl}\, H_G$ is compact and *0-dimensional* if the image of $\mathrm{Cl}\, H_G$ is 0-dimensional in E^n/G.

Theorem 3. *Let G and F be compact, 0-dimensional decompositions of E^m and E^n respectively, each element of which possesses property* UV^∞. *Then* $\mathrm{Cl}\,(H_G \times H_F)$ *is definable by cells in* $E^m \times E^n$.

Corollary. $H_G \times H_F$ *is shrinkable in* $E^m \times E^n$.

The proof of Theorem 3 requires the following lemma which is a simple generalization of Lemma 1 in [7].

Lemma 8. *Suppose that $M_1 \subset M_2 \subset \cdots \subset M_{k-r+1}$ is a sequence of finite combinatorial k-manifolds (not necessarily connected) such that each M_i is a combinatorial subspace of M_{i+1} and the inclusion of each component of M_i into M_{i+1} is homotopically trivial. If Y is any subcomplex of M_1 such that* $\dim Y \leq k - r - 1$ *and $r \geq 2$, then Y lies in a finite union of disjoint k-cells in M_{k-r+1}.*

In the case $r = 2$, any subcomplex of M_1 having codimension 3 lies in the union of k-cells in M_{k-1}. Suppose that $M_1 \supset M_2 \supset \cdots$ is a sequence of k-manifolds such that any subcomplex $Y \subset M_{i+1}$ having codimension 3 lies in a finite number of disjoint k-cells in M_i. We will call $\{M_i\}$ a *special sequence.*

Corollary. *If $\{M_i\}$ is a sequence of finite combinatorial k-manifolds such that each M_{i+1} is a combinatorial subspace of M_i and the inclusion of each component of M_{i+1} into M_i is homotopically trivial, then $\{M_i\}$ can be refined to a special sequence.*

As in [1], the sequence $H_1, H_2, \cdots$ of compact m-manifolds-with-boundary will be called a *defining sequence* for the decomposition G of E^m provided $H_{i+1} \subset \mathrm{Int}\, H_i$ for each i, and g is a nondegenerate element of G if and only if g is a nondegenerate component of $\bigcap_{i=1}^{\infty} H_i$. We will say that H_i has a k-spine if H_i is PL and H_i collapses to a subpolyhedron of dimension k or less.

Proof of Theorem 3. Using the techniques in [1], there exist PL defining sequences $\{H_i\}$ and $\{K_i\}$ for G and F respectively, such that

(i) Each H_{i+1} (K_{i+1}) is a finite combinatorial subspace of H_i (K_i).

(ii) The inclusion of each component of H_{i+1} (K_{i+1}) into H_i (K_i) is homotopically trivial.

(iii) Each H_i (K_i) collapses to a spine of codimension 2.

Consider $\{H_i \times K_i\}$ which is a PL defining sequence for the decomposition $G \times F$ of $E^m \times E^n$. Note the elements of $G \times F$ are $\{\alpha \times \beta \mid \alpha \in G, \beta \in F\}$. It is easy to check that:

(i) Each $H_i \times K_i$ is a finite combinatorial $m+n$-manifold.

(ii) The inclusion of each component of $H_{i+1} \times K_{i+1}$ into $H_i \times K_i$ is homotopically trivial.

(iii) Each $H_i \times K_i$ collapses to a spine of codimension four.

(iv) W.l.o.g., $\{H_i \times K_i\}$ is a special sequence.

If we let $M_i = H_i \times K_i$ and let M_i' be the spine, it follows that there exist finitely many disjoint $m+n$-cells $B_1, \cdots, B_k$ such that $M_{i+1}' \subset \bigcup_{i=1}^k B_i \subset M_i$. Thus there exists a PL homeomorphism, as in [16], $b\colon E^m \times E^n \to E^m \times E^n$ which is fixed outside of M_i such that

$$M_{i+1} \subset b\left(\bigcup_{i=1}^{k} B_i\right) \subset M_i.$$

Since $H_G \times H_F = \bigcap_{i=1}^{\infty} M_i$, it follows that $H_G \times H_F$ is definable by cells, which concludes the proof of Theorem 3.

In [15], Siebenmann points out that Bing's criterion (shrinkability) is a necessary condition for the existence of a homeomorphism $E^n/G \times E^m \cong E^{m+n}$ ($n + m > 4$), at least provided the elements of G have property UV^{∞}.

Theorem 4. *Let G and H be compact, 0-dimensional decompositions of E^m and E^n respectively, such that each element of G (and H) has property* UV^{∞}. *If $E^m/G \times E^n \cong E^{m+n}$, $E^n/H \times E^m \cong E^{m+n}$, then $E^m/G \times E^n/H \cong E^{m+n}$.*

Proof. If $m+n > 4$, then Siebenmann's result mentioned above [15] together with Theorem 3 imply Theorem 4. If $m = 3$ and $n = 1$, then the theorem is trivially a consequence of the hypothesis. If $m = n = 2$, then results of Moore imply the theorem.

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